

# Supersymmetric oscillator: novel symmetries

R. Kumar<sup>(a)\*</sup> and R. P. Malik<sup>(a,b)†</sup>

<sup>(a)</sup>*Physics Department, Centre of Advanced Studies,  
Banaras Hindu University, Varanasi - 221 005, India*

<sup>(b)</sup>*DST Centre for Interdisciplinary Mathematical Sciences,  
Faculty of Science, Banaras Hindu University, Varanasi - 221 005, India*

We discuss various continuous and discrete symmetries of the supersymmetric simple harmonic oscillator (SHO) in one  $(0 + 1)$ -dimension of spacetime and show their relevance in the context of mathematics of differential geometry. We show the existence of a novel set of discrete symmetries in the theory which has, hitherto, not been discussed in the literature on theoretical aspects of SHO. We also point out the physical relevance of our present investigation.

PACS numbers: 11.30.Pb, 03.65.-w, 02.40.-k

## I. INTRODUCTION

The model of a harmonic oscillator (HO) is one of the most studied models in the history of theoretical physics. This is mainly due to the fact that it is an exactly solvable cute model which encompasses in its folds a rich and elegant mathematical structure. Theoretically, this model has been able to explain a multitude of phenomena in diverse domains of physics as well as other key branches of science. A supersymmetric version of the above HO, that incorporates the bosonic and fermionic variables, provides a prototype example of supersymmetry and its innate supersymmetric algebra [1, 2].

The purpose of our present investigation is to discuss the discrete and continuous symmetries of the one  $(0 + 1)$ -dimensional (1D) supersymmetric harmonic oscillator (SHO) and establish that it is a cute 1D model for the Hodge theory. In fact, the symmetries of the SHO provide a physical realization of the de Rham cohomological operators of differential geometry [3, 4]. Whereas the continuous symmetries (and corresponding generators) provide the analogue of the de Rham cohomological operators, the discrete symmetry of this model corresponds to the Hodge duality operation of differential geometry. Thus, the SHO is a complete model for the Hodge theory.

In our earlier set of works [5–7], we have shown that the 1-form and 2-form gauge field theories (in two  $(1+1)$ -dimensions and four  $(3 + 1)$ -dimensions of spacetime) provide physical models for the Hodge theory. We have also discussed a toy model of a 1D rigid rotor and demonstrated that it provides a cute model for the Hodge theory (at the algebraic level) [7]. In all the above models, it is the existence of continuous and discrete symmetry transformations (and their corresponding generators) that provide the physical realizations of the abstract mathematical objects of differential geometry.

It is worthwhile to mention that the above kind of studies have enabled us to prove that the 2D Abelian 1-

form and 4D Abelian 2-form gauge theories provide a new model for the topological field theory [8] and an example of the quasi-topological field theory [9], respectively. Thus, such kind of studies are physically interesting.

All the above models are, however, based on the gauge symmetries that are generated by the first-class constraints in the language of Dirac's prescription for the classification scheme of constraints [10]. So far, we have *not* studied a supersymmetric model in the purview of mathematical structure of a Hodge theory. In our present investigation, we try to accomplish this goal by taking the example of SHO where the physical continuous and discrete symmetries of the theory play very important roles. We plan to extend our present work to other superpotentials (e.g. shape-invariant potentials) which have been shown to be of some physical interests [2].

The following factors have contributed to our curiosity in pursuing our present investigation. First and foremost, the model of SHO is one of the prototype examples of a supersymmetric theory which has been studied from many different angles. Thus, it is always challenging to state something *new* about this model. Second, to prove a model to be an example of the Hodge theory, one has to examine and explore various kinds of symmetries so that the abstract mathematical quantities, associated with the Hodge theory, could be explained in terms of the symmetry transformations. This is *an uphill* task. Thus, we are highly motivated to accomplish this goal with the sophistication of theoretical physics. Finally, the present model is *not* a gauge field theoretic model. Thus, the model of SHO is unique in its own right because, even though, it is a model for the Hodge theory, it does not lean heavily on the idea of Dirac's first-class constraints and associated local and continuous gauge symmetries.

The contents of our present paper are organized as follows. To set up the notations and conventions, we recapitulate the bare essentials of the Lagrangian and Hamiltonian formulations of the SHO in section two. Our section three deals with the existence of a couple of nilpotent continuous symmetry transformations plus a bosonic symmetry transformation. The latter is obtained from the above nilpotent symmetry transformations. Our section four is devoted to the description of discrete symme-

---

\*Electronic address: raviphynuc@gmail.com

†Electronic address: malik@bhu.ac.in

tries in the theory. We deduce the algebraic structures of the symmetry transformations (and corresponding generators) in section five. Finally, we make some concluding remarks in section six.

## II. PRELIMINARIES: CANONICAL FORMALISM

We begin with the Lagrangian for the 1D SHO with unit mass (i.e.  $m = 1$ ) and natural frequency  $\omega$ . This physical system is described by the ordinary and Grassmannian dynamical variables at the classical level. The explicit form of the Lagrangian is (see, e.g. [11])

$$L = \frac{1}{2} \dot{x}^2 - \frac{1}{2} \omega^2 x^2 + i \bar{\psi} \dot{\psi} - \omega \bar{\psi} \psi, \quad (1)$$

where  $\dot{x} = (dx/dt)$  and  $\dot{\psi} = (d\psi/dt)$  are the generalized velocities of the SHO in terms of its instantaneous position  $x$ , Grassmannian variable  $\psi$  and the evolution parameter  $t$ . Here the pair  $(\psi, \bar{\psi})$  are the Grassmannian variables (with  $\psi^2 = 0, \bar{\psi}^2 = 0, \psi\bar{\psi} + \bar{\psi}\psi = 0$ ) and we adopt the convention of left derivative for these objects. As a consequence, we obtain the following Hamiltonian by exploiting the Legendre transformation, namely;

$$\begin{aligned} H &= \dot{\psi} \Pi_{\psi} + \dot{\bar{\psi}} \Pi_{\bar{\psi}} + \dot{x} p - L \\ &= \frac{1}{2} p^2 + \frac{1}{2} \omega^2 x^2 + \omega \bar{\psi} \psi, \end{aligned} \quad (2)$$

where  $p = (\partial L / \partial \dot{x}) = \dot{x}$ ,  $\Pi_{\psi} = (\partial L / \partial \dot{\psi}) = -i\bar{\psi}$ ,  $\Pi_{\bar{\psi}} = (\partial L / \partial \dot{\bar{\psi}}) = 0$  are the canonical conjugate momenta corresponding to the variables  $x, \psi$ , and  $\bar{\psi}$ .

We can define the following bosonic and fermionic creation and annihilation operators in terms of the suitable dynamical variables of the Lagrangian. These operators (with  $\hbar = c = 1$  and  $m = 1$ ) are (see, e.g. [11])

$$\begin{aligned} a_B^\dagger &= \frac{1}{\sqrt{2\omega}} (-i p + \omega x), & a_F^\dagger &= \bar{\psi}, \\ a_B &= \frac{1}{\sqrt{2\omega}} (i p + \omega x), & a_F &= \psi. \end{aligned} \quad (3)$$

In view of the above, we can easily check that

$$H = \omega (a_B^\dagger a_B + a_F^\dagger a_F) \equiv \frac{1}{2} p^2 + \frac{1}{2} \omega^2 x^2 + \omega \bar{\psi} \psi. \quad (4)$$

We can verify, in a straightforward manner, that the following operators

$$N_B = a_B^\dagger a_B, \quad N_F = a_F^\dagger a_F, \quad Q = a_B^\dagger a_F, \quad \bar{Q} = a_F^\dagger a_B, \quad (5)$$

are the conserved quantities because they commute with the Hamiltonian of the theory if we use the following basic commutator and anticommutator brackets

$$[a_B, a_B^\dagger] = 1, \quad \{a_F, a_F^\dagger\} = 1, \quad (6)$$

and take all the rest of the brackets to be zero. In other words, we take  $(a_F^\dagger)^2 = \frac{1}{2} \{a_F^\dagger, a_F^\dagger\} = 0, a_F^2 = \frac{1}{2} \{a_F, a_F\} = 0, [a_B, a_F] = 0, [a_F^\dagger, a_B^\dagger] = 0$ , etc. Exploiting these brackets, it can be proved that the conserved fermionic and bosonic quantities  $Q, \bar{Q}, N_B, N_F$  and  $H$  obey the following explicit algebra

$$\begin{aligned} [Q, H] &= [\bar{Q}, H] = 0, & [N_B, H] &= [N_F, H] = 0, \\ Q^2 &= \frac{1}{2} \{Q, Q\} = 0, & \bar{Q}^2 &= \frac{1}{2} \{\bar{Q}, \bar{Q}\} = 0, \\ \{Q, \bar{Q}\} &= \frac{H}{\omega}, & [Q, N_B] &= -Q, \quad [\bar{Q}, N_B] = \bar{Q}, \\ [Q, N_F] &= Q, & [\bar{Q}, N_F] &= -\bar{Q}, \end{aligned} \quad (7)$$

which shows that the Hamiltonian  $H$  of the theory is the Casimir operator for the whole algebra.

We wrap up this section with the remarks that the following fermionic ( $Q^2 = \bar{Q}^2 = 0$ ) conserved quantities (i.e.  $\dot{Q} = -i[Q, H] = 0, \dot{\bar{Q}} = -i[\bar{Q}, H] = 0$ ), expressed in terms of the dynamical variables, namely;

$$Q = \frac{1}{\sqrt{2\omega}} (-ip + \omega x) \psi, \quad \bar{Q} = \frac{1}{\sqrt{2\omega}} \bar{\psi} (ip + \omega x), \quad (8)$$

can be derived from the nilpotent continuous symmetries of the Lagrangian (1) as the Noether conserved charges. Similarly, the Hamiltonian  $H$  [cf. (4)] can also be derived as a conserved charge corresponding to a continuous bosonic symmetry (that is obtained from the above nilpotent symmetries). We discuss these continuous symmetry transformations in our next section.

## III. CONTINUOUS SYMMETRIES

It is interesting to note that under the following infinitesimal, local, continuous and nilpotent (i.e.  $s_1^2 = 0, s_2^2 = 0$ ) transformations

$$\begin{aligned} s_1 x &= \frac{-i \psi}{\sqrt{2\omega}}, & s_1 \bar{\psi} &= \frac{1}{\sqrt{2\omega}} (\dot{x} + i \omega x), & s_1 \psi &= 0, \\ s_2 x &= \frac{i \bar{\psi}}{\sqrt{2\omega}}, & s_2 \psi &= \frac{1}{\sqrt{2\omega}} (-\dot{x} + i \omega x), & s_2 \bar{\psi} &= 0, \end{aligned} \quad (9)$$

the Lagrangian of SHO transforms as:

$$s_1 L = \frac{d}{dt} \left( -\frac{\omega}{\sqrt{2\omega}} x \psi \right), \quad s_2 L = \frac{d}{dt} \left( \frac{i}{\sqrt{2\omega}} \dot{x} \bar{\psi} \right). \quad (10)$$

As a consequence, the action remains invariant under the continuous transformations  $s_1$  and  $s_2$ . Now, there are two side remarks in order. First, the symmetry transformations  $s_1$  and  $s_2$  are nilpotent of order two (i.e.  $s_1^2 = 0, s_2^2 = 0$ ) only on the *on-shell* where the equations of motion  $\dot{\psi} + i \omega \psi = 0$  and  $\dot{\bar{\psi}} - i \omega \bar{\psi} = 0$  are valid. Second, the fermionic transformations  $s_1$  and  $s_2$  commute with the bosonic pair  $(x, p)$  and anticommute with the fermionic pair  $(\psi, \bar{\psi})$ . The above inputs are important for our rest of the discussions in our present endeavor.

Using the Noether's theorem, it is straightforward to verify that the conserved charges (8) emerge from the nilpotent continuous symmetry transformations (9). In other words, the charges (8) are the generators of the symmetry transformations (9) [for the Lagrangian (1) of SHO] because

$$s_1 \Phi = \pm i [\Phi, Q]_{\pm}, \quad s_2 \Phi = \pm i [\Phi, \bar{Q}]_{\pm}, \quad (11)$$

where (+)– signs on the square bracket correspond to the (anti)commutator for the generic dynamical variable  $\Phi = x, \psi, \bar{\psi}$  being (fermionic) bosonic in nature. Furthermore, depending on the (fermionic) bosonic nature of  $\Phi$ , we choose the (+)– signs, that are present, in front of the square brackets in equation (11).

The nilpotent ( $s_1^2 = 0, s_2^2 = 0$ ) transformations  $s_1$  and  $s_2$  do not anticommute (i.e.  $\{s_1, s_2\} \neq 0$ ). As a consequence, we define a bosonic symmetry transformation  $s_W = \{s_1, s_2\}$  (i.e.  $s_W^2 \neq 0$ ), under which, the dynamical variables ( $x, \psi, \bar{\psi}$ ) transform as given below:

$$\begin{aligned} s_W x &= \frac{i}{\omega} \dot{x}, & s_W \psi &= \frac{i}{2\omega} (\dot{\psi} - i\omega\psi), \\ s_W \bar{\psi} &= \frac{i}{2\omega} (\dot{\bar{\psi}} + i\omega\bar{\psi}). \end{aligned} \quad (12)$$

The above transformations are *symmetry* transformations because the Lagrangian  $L$  [cf. (1)] transforms to a total time derivative:

$$s_W L = \frac{d}{dt} \left[ \frac{i}{2\omega} (\dot{x}^2 - \omega^2 x^2 + i\bar{\psi}\dot{\psi} - \omega\bar{\psi}\psi) \right]. \quad (13)$$

As a consequence, the action integral  $S = \int dt L$  remains invariant for the physically well-defined dynamical variables that are present in the theory.

According to Noether's theorem, we have the following expression for the conserved charge ( $W$ ) corresponding to the transformations  $s_W$  in (12):

$$\begin{aligned} W &= \frac{i}{\omega} \left( \frac{1}{2} \dot{x}^2 + \frac{1}{2} \omega^2 x^2 + \omega \bar{\psi} \dot{\psi} \right) \\ &\equiv \frac{i}{\omega} \left( \frac{p^2}{2} + \frac{1}{2} \omega^2 x^2 + \omega \bar{\psi} \dot{\psi} \right). \end{aligned} \quad (14)$$

Thus, we note that the above conserved charge [i.e.  $W = (i/\omega)H$ ] is connected with the Hamiltonian  $H$  of SHO. As a consequence, basically  $H$  is the generator of the infinitesimal, local and continuous transformations (12). In reality, it is elementary to verify that  $s_W \Phi = -i[\Phi, W]$  for  $\Phi = x, \psi, \bar{\psi}$  when one uses the equations of motion  $\dot{\psi} + i\omega\psi = 0, \dot{\bar{\psi}} - i\omega\bar{\psi} = 0$  derived from (1).

We close this section with the remark that the conserved operators  $N_B = a_B^\dagger a_B$  and  $N_F = a_F^\dagger a_F$ , expressed in terms of the dynamical variables, also generate continuous symmetry transformations for the Lagrangian (1). However, the ensuing symmetries are incorporated in the symmetry generated by  $W = (i/\omega)H \equiv (i/\omega)(a_B^\dagger a_B + a_F^\dagger a_F)$ . This is precisely the reason that these symmetries have not been discussed separately and independently in our present discussion.

#### IV. DISCRETE SYMMETRIES

The Lagrangian  $L$  of SHO also respects the following discrete symmetry transformations

$$\psi \rightarrow \pm i \bar{\psi}, \quad \bar{\psi} \rightarrow \mp i \psi, \quad x \rightarrow -x, \quad \omega \rightarrow -\omega. \quad (15)$$

It is straightforward to check that the Lagrangian ( $L$ ) remains invariant (i.e.  $L \rightarrow L$ ) under the above discrete transformations. The transformations (15) are the analogue of the Hodge duality operation of differential geometry. To corroborate the above statement, first of all, it can be verified that two successive transformations, corresponding to (15), on the dynamical variables of the theory (i.e.  $x, \psi, \bar{\psi}$ ) lead to the following [12]:

$$* (* \psi) = + \psi, \quad * (* \bar{\psi}) = + \bar{\psi}, \quad * (* x) = + x, \quad (16)$$

where  $(*)$  is the discrete symmetry transformations listed in (15). Thus, we note that all the dynamical variables (i.e.  $x, \psi, \bar{\psi}$ ) acquire positive signature under the operations of two successive discrete transformations (15). We shall exploit this observation later on.

The nilpotent and continuous symmetry transformations  $s_1$  and  $s_2$  are connected by the following interesting relationship:

$$s_2 \Phi = \pm * s_1 * \Phi, \quad \Phi = x, \psi, \bar{\psi}, \quad (17)$$

where the (+)– signs on the r.h.s. of (17) are dictated by the signatures that are present in the relationship (15). One can also check that

$$s_1 \Phi = \mp * s_2 * \Phi, \quad \Phi = x, \psi, \bar{\psi}, \quad (18)$$

where we re-emphasize that the  $(*)$ , in equations (17) and (18), corresponds to the discrete transformations of equation (15). The difference of signatures in (17) and (18) do crop up in theories with duality symmetry [12].

We note that the relations in (17) and (18) are reminiscent of the relationship between the co-exterior derivative ( $\delta = \pm * d*$ ) and the exterior derivative ( $d = dx^\mu \partial_\mu$ ) of differential geometry which are also nilpotent of order two (i.e.  $d^2 = 0, \delta^2 = 0$ ). Thus, the Hodge duality  $(*)$  operation of differential geometry is reflected in the existence of a set of discrete symmetry transformations (15) for SHO. We also note that the duality operation  $(*)$  on the fermionic conserved charges, under (15), yields

$$\begin{aligned} * Q &= \pm \bar{Q}, & * \bar{Q} &= \mp Q, \\ * (* Q) &= -Q, & * (* \bar{Q}) &= -\bar{Q}. \end{aligned} \quad (19)$$

Thus, the conserved charges  $Q$  and  $\bar{Q}$  transform as:  $Q \rightarrow \pm \bar{Q}, \bar{Q} \rightarrow \mp Q$  which is like the duality transformations in the electrodynamics where we have:  $\vec{E} \rightarrow \pm \vec{B}, \vec{B} \rightarrow \mp \vec{E}$  for the electric and magnetic fields present in the source-free Maxwell's equations. Further, it should be noted that two successive operations of discrete transformations (15) on  $Q$  and  $\bar{Q}$  lead to  $(-)$  sign on the r.h.s. This means that our present result is totally opposite

to the effect of such operations on the dynamical variables  $x, \psi, \bar{\psi}$  [cf. (16)]. Finally, we find that (with  $*H = H$ ,  $*W = -W$ ,  $*N_B = -N_B$ ,  $*N_F = -N_F$ ), the total algebra amongst the operators  $Q, \bar{Q}, W, N_B, N_F$  remains invariant under the discrete symmetry transformations (15) which correspond to the  $(*)$  operation.

## V. ALGEBRAIC STRUCTURES

One can very clearly verify that the continuous symmetry transformations  $(s_1, s_2, s_W)$  of **section three** satisfy the following operator algebra:

$$\begin{aligned} s_1^2 = 0, \quad s_2^2 = 0, \quad s_W = \{s_1, s_2\} = (s_1 + s_2)^2, \\ [s_1, s_W] = 0, \quad [s_2, s_W] = 0, \quad s_W^2 \neq 0. \end{aligned} \quad (20)$$

The above algebraic structures are true on the on-shell condition where equations of motion  $\ddot{x} + \omega^2 x = 0$ ,  $\dot{\bar{\psi}} - i\omega\psi = 0$ ,  $\dot{\psi} + i\omega\bar{\psi} = 0$  are satisfied. The algebra (20) is reminiscent of the algebra obeyed by the de Rham cohomological operators  $(d, \delta, \Delta)$  of differential geometry. The latter algebra, for the cohomological operators, can be succinctly expressed as [3, 4]

$$\begin{aligned} d^2 = 0, \quad \delta^2 = 0, \quad \{d, \delta\} = \Delta = (d + \delta)^2, \\ [\Delta, d] = 0, \quad [\Delta, \delta] = 0, \quad \delta = \pm * d *, \end{aligned} \quad (21)$$

where  $\Delta$  is the Casimir operator and  $(*)$  is the Hodge duality operation on a given spacetime manifold. The  $(+)$ - signs in  $\delta = \pm * d *$  are determined by the inner product of specific forms (see, e.g. [3, 4] for details). For our present model of SHO, we have already shown the analogue of relation  $(\delta = \pm * d *)$  in equations (17) and (18) for the generic dynamical variable  $\Phi$  of the theory.

One of the key properties of the exterior derivative  $d$  is the fact that when it operates on a  $n$ -form  $(f_n)$  it raises its degree by one (i.e.  $df_n \sim f_{n+1}$ ). Similarly, when  $\delta$  acts on  $f_n$ , it lowers its degree by one (i.e.  $\delta f_n \sim f_{n-1}$ ). In contrast to  $d$  and  $\delta$ , the Laplacian operator  $\Delta$  does not change the degree of a  $n$ -form  $(f_n)$ , on which, it operates (i.e.  $\Delta f_n \sim f_n$ ). These properties are very sacrosanct in the context of cohomological discussions.

The above properties are also captured within the purview of the symmetry considerations and conserved charges. To verify it, let us define the eigen state  $|n_B\rangle$  with respect to the number operator  $N_B$  (i.e.  $N_B|n_B\rangle = n_B|n_B\rangle$ ). Using the algebraic relations in (7), it can be explicitly checked that

$$\begin{aligned} N_B Q |n_B\rangle &= (n_B + 1) Q |n_B\rangle, \\ N_B \bar{Q} |n_B\rangle &= (n_B - 1) \bar{Q} |n_B\rangle, \\ N_B W |n_B\rangle &= n_B W |n_B\rangle. \end{aligned} \quad (22)$$

Thus, the eigen values of the states  $Q|n_B\rangle, \bar{Q}|n_B\rangle$  and  $W|n_B\rangle$ , w.r.t. the operator  $N_B$ , are  $(n_B + 1)$ ,  $(n_B - 1)$  and  $n_B$ , respectively. This observation is similar to the consequences that ensue from the operation of  $d, \delta, \Delta$  on

a given form of degree  $(n_B)$ . Thus, the analogy between the de Rham cohomological operators  $(d, \delta, \Delta)$  and the conserved charges  $(Q, \bar{Q}, W)$  of our present theory is true.

The cohomological properties of  $(d, \delta, \Delta)$  can also be captured in terms of the fermionic number  $n_F$  if we start with the eigen state  $|n_F\rangle$  satisfying  $N_F|n_F\rangle = n_F|n_F\rangle$ . It is evident, from equations (5) and (7), that

$$\begin{aligned} N_F \bar{Q} |n_F\rangle &= (n_F + 1) \bar{Q} |n_F\rangle, \\ N_F Q |n_F\rangle &= (n_F - 1) Q |n_F\rangle, \\ N_F W |n_F\rangle &= n_F W |n_F\rangle. \end{aligned} \quad (23)$$

We note, from the above, that the eigen values of the states  $\bar{Q}|n_F\rangle, Q|n_F\rangle$  and  $W|n_F\rangle$ , w.r.t. the operator  $N_F$ , are  $(n_F + 1)$ ,  $(n_F - 1)$  and  $n_F$ , respectively. Thus, we conclude that there are two realizations of  $(d, \delta, \Delta)$  in the language of symmetry generators  $(Q, \bar{Q}, W)$ . If the degree of a form is identified with the bosonic number  $n_B$ , we have the following mapping:

$$(d, \delta, \Delta) \iff (Q, \bar{Q}, W). \quad (24)$$

On the other hand, when the degree of a differential form is identified with  $n_F$ , then, the operations of  $(d, \delta, \Delta)$  and symmetry generators lead to one-to-one correspondence as given below:

$$(d, \delta, \Delta) \iff (\bar{Q}, Q, W). \quad (25)$$

Thus, the algebraic structures of (7), in terms of the conserved operators, capture the algebraic structures of (21). As a consequence, our present model (i.e. SHO) is a prototype example of the Hodge theory.

## VI. CONCLUSIONS

In our present investigation, we have shown that the Lagrangian (1) of SHO respects *three* continuous symmetry transformations (i.e.  $s_1, s_2$  and  $s_W$ ). These continuous symmetries (and their corresponding generators) provide a physical realization of the de Rham cohomological operators of differential geometry at the algebraic level. In addition to the above continuous symmetries, the Lagrangian (1) also respects discrete symmetries which correspond to the Hodge duality operation  $(*)$  of differential geometry. This symmetry enables us to establish the appropriate relations  $(s_2 = \pm * s_1 *, s_1 = \mp * s_2 *)$  between the transformations  $s_1$  and  $s_2$ . These relations are exactly same as the relation  $(\delta = \pm * d *)$  between  $\delta$  and  $d$  of the differential geometry.

In addition to the above, the generator  $Q$  (corresponding to the transformations  $s_1$ ) increases the bosonic number  $n_B$  by one unit and decreases the fermionic number  $n_F$  by one unit whereas the generator  $\bar{Q}$  (corresponding to the symmetry transformations  $s_2$ ) does its opposite. The bosonic generator  $W$  does not alter  $n_B$  as well as  $n_F$ . It is interesting to point out that there is a *one-to-one* correspondence between the generators  $(Q, \bar{Q}, W)$

and de Rham cohomological operators ( $d, \delta, \Delta$ ) of differential geometry. The algebra obeyed by the symmetry transformations (and their corresponding generators) is reminiscent of the algebra of de Rham cohomological operators of differential geometry. Thus, the present theory provides a cute model for the Hodge theory.

The above Hodge algebra has also been obtained in the case of 2D free Abelian 1-form as well as 4D free Abelian 2-form gauge theories [5–7]. In these theories, the nilpotent ( $Q_{a(b)}^2 = 0$ ) (anti-)BRST charges  $Q_{a(b)}$  (corresponding to the (anti-)BRST symmetry transformations  $s_{a(b)}$ ) increase and decrease the ghost number by one unit, respectively. The nilpotent ( $Q_{a(d)}^2 = 0$ ) (anti-)co-BRST charges (corresponding to the (anti-)co-BRST symmetry transformations  $s_{a(d)}$ ) have an opposite effect on the ghost number. The bosonic symmetry, obtained from the anticommutator of the above nilpotent symmetries, does not increase or decrease the ghost number. Therefore, there is *two-to-one* mapping between the continuous symmetry transformations (and their corresponding generators) and the de Rham cohomological operators of differential geometry (see, e.g. [5–7] for details).

It is clear from the above discussions that the symmetry transformations (and their corresponding generators) of SHO have similar kind of algebra as obeyed by the symmetry transformations (and their corresponding generators) of the 2D free Abelian 1-form as well as 4D free Abelian 2-form gauge theories. However, there is a glaring difference as far as physical realizations of the cohomological operators are concerned. Whereas there is two-to-one mapping between the conserved charges and the cohomological operators within the framework of BRST formalism, there is one-to-one mapping between the conserved charges and cohomological operators in the case of SHO. Physically, we feel, the model of SHO is more

appealing than the BRST analysis of gauge theories because the number operators ( $n_F$  and  $n_B$ ) are more physical than the ghost number in the BRST approach. We plan to extend our present work to other superpotentials that have been discussed in [2]. In particular, we strongly feel that the theories with shape-invariant superpotentials would also be models for the Hodge theory.

In the existing literature on supersymmetric algebra and cohomological discussions [13–16], a whole lot of developments have been made on the (hyper)Kähler manifolds. One dimensional supersymmetric structures have been found for the twisted and untwisted versions of the super Yang-Mills theory [15]. However, our present model of SHO is a simple case where the properties of the (super)Kähler manifolds are *not* invoked at all. The model of SHO is one of the simplest supersymmetric system that provides the physical realizations of cohomological operators in terms of suitable symmetries.

The analogue of the Laplacian operator ( $\Delta$ ) of differential geometry is the continuous symmetry transformation (12) for SHO. However, as far as the basic tenet of supersymmetric quantum mechanics is concerned, the anticommutator  $\{s_1, s_2\}$  should lead to the time translation (i.e.  $\{s_1, s_2\}\Phi \sim \dot{\Phi}$ ). It can be verified that the transformations (12) satisfy this requirement, too, when we use the appropriate equations of motion. To be precise, it turns out that  $\{s_1, s_2\}\Phi = (i/\omega)\dot{\Phi}$  for the generic dynamical variable  $\Phi = x, \psi, \bar{\psi}$  if we use the equations of motion  $\dot{\psi} + i\omega\psi = 0$  and  $\dot{\bar{\psi}} - i\omega\bar{\psi} = 0$  [derived from the Lagrangian (1)]. Thus, our present model of SHO provides a complete and cute model for the Hodge theory.

**Acknowledgement:** One of us (RK) would like to thank UGC, Government of India, New Delhi, for financial support under the JRF scheme.

- 
- [1] See, e.g., E. Witten, Nucl. Phys. B **185**, 531 (1981).
  - [2] See, e.g., for an excellent review, F. Cooper, A. Khare, U. Sukhatme, Phys. Rep. **251**, 264 (1995).
  - [3] See, e.g., S. Mukhi, N. Mukanda, *Introduction to Topology, Differential Geometry and Group Theory for Physicists* (Wiley Eastern Private Limited, New Delhi, 1990).
  - [4] T. Eguchi, P. B. Gilkey, A. Hanson, Phys. Rep. **66**, 213 (1980).
  - [5] R. P. Malik, Mod. Phys. Lett. A **15**, 2079 (2000).
  - [6] S. Gupta, R. P. Malik, Eur. Phys. J. C **58**, 517 (2008).
  - [7] S. Gupta, R. P. Malik, Eur. Phys. J. C **68**, 325 (2010).
  - [8] R. P. Malik, J. Phys. A: Math. Gen. **41**, 4167 (2001).
  - [9] R. P. Malik, J. Phys. A: Math. Gen. **36**, 5095 (2003).
  - [10] P. A. M. Dirac, *Lectures on Quantum Mechanics* [Belfer Graduate School of Science] (Yeshiva University Press, New York, 1964).
  - [11] See, e.g., for an excellent exposition, A. Das, *Field Theory: A Path Integral Approach* (World Scientific, Singapore, 1993).
  - [12] S. Deser, A. Gomberoff, M. Henneaux, C. Teitelboim, Phys. Lett. B **400**, 80 (1997).
  - [13] J. M. Figueroa-O'Farrill, C. Koehl, B. Spence, Nucl. Phys. B **503**, 614 (1997).
  - [14] R. Zucchini, Class. Quant. Grav. **24**, 2073 (2007).
  - [15] L. Baulieu, F. Toppan, Lett. Math. Phys. **98**, 299 (2011).
  - [16] F. Brandt, J. Math. Phys. **51**, 122302 (2010).